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General solution of Kostin's equation arising in quantum mechanics

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Abstract. For the non-linear ordinary differential equation in the radial component of the probability density discussed by Kostin, a general solution is obtained. This solution is a homogeneous function of two linearly independent functions satisfying a certain associated linear homogeneous equation.

1. Introduction

From the Schrödinger equation for a complex wavefunction, Kostin [1] derives the following non-linear differential equation for the probability density $P(r, \theta, \phi)$ in quantum theory:

$$P\nabla^2 P=(4m/\hbar^2)(V-E)P^2+\tfrac{1}{2}\nabla P\cdot\nabla P+(2m^2/\hbar^2)\boldsymbol{J}\cdot\boldsymbol{J} \tag{1.1a}$$

where the probability current density vector is given by

$$\boldsymbol{J}=(\mathrm{i}\hbar/2m)[\psi\nabla\psi^*-\psi^*\nabla\psi]. \tag{1.1b}$$

Applying the above result for the motion of a particle in a centrally symmetric field, he further obtains the second-order non-linear ordinary differential equation in $R(r)$:

$$L_1 R=\tfrac{1}{2}R'^2R^{-1}+(2m^2/\hbar^2)J_r^2R^{-1} \qquad R'=\mathrm{d}R/\mathrm{d}r \tag{1.2}$$

where $R(r)$ and $J_r(r)$ are the components of the probability density and the probability current density vector, respectively, in the radial direction, L, $\hbar$ and m are constants, E is a degenerate eigenvalue of the Hamiltonian operator of the Schrödinger equation and L_k represents the linear differential operator

$$L_k=\frac{\mathrm{d}^2}{\mathrm{d}r^2}+\frac{2}{r}\frac{\mathrm{d}}{\mathrm{d}r}-k\left(\frac{2L(L+1)}{r^2}+\frac{4m}{\hbar^2}(V(r)-E)\right). \tag{1.3}$$

When $J_r=0$ or the wavefunction is real, the equation corresponding to (1.2) is

$$L_1 R=\tfrac{1}{2}R'^2R^{-1}. \tag{1.4}$$

For the case $J_r\neq 0$, noting that $r^2J_r(r)$ is constant, Kostin introduces the radial distribution function $\rho(r)=4\pi r^2R(r)$ as the new dependent variable and discusses the solutions of (1.2) and (1.4) in terms of the solutions of equivalent third-order linear differential equations in $\rho(r)$ and $R(r)$, respectively.

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The aim of this paper is to find the general solutions of (1.2) and (1.4) as real functions of two linearly independent solutions of the associated linear homogeneous equation

$$L_{1/2}R = 0 \tag{1.5}$$

which is the radial Schrödinger equation. We note that equation (1.2) is analogous to the one which we would have to solve if, in a one-dimensional problem, a particle of mass m were to move with an appropriate effective potential.

It is well known in scattering theory that the phase shift is given in terms of the logarithmic derivative β_l of the solution of (1.5). It may be pointed out here that the constant $r^2 J_r$ is closely connected with β_l (see, for example, [2]).

2. Some preliminary results

The author [3] has discussed recently the solutions of some classes of second-order non-linear ordinary differential equations of the form

$$\ddot{y} + p(t)\dot{y} + q(t)y = \mu\dot{y}^2 y^{-1} + f(t)y^n \qquad \mu \neq 1, n \neq 1 \tag{2.1}$$

where $p(t)$ and $q(t)$ are known functions and the dots denote differentiation with respect to the independent variable t. The solutions are obtained as homogeneous functions of two linearly independent solutions $u(t)$ and $v(t)$ of the associated linear equation

$$\ddot{y} + p(t)\dot{y} + (1-\mu)q(t)y = 0 \tag{2.2}$$

under different constraints on the function $f(t)$. If $W(u, v) = \dot{u}v - u\dot{v}$ is the Wronskian and

$$F(t) = \int^t p(t)\,\mathrm{d}t \tag{2.3}$$

then we have the following result due to Abel [4]:

$$W(u, v)\exp(F(t)) = \text{constant} = C_w(\text{say}). \tag{2.4}$$

C_w is a definite non-zero constant, called the Abel constant, once the solutions u and v of (2.2) are chosen.

We now recall the following two theorems from [3].

Theorem 1. The general solution of the equation

$$\ddot{y} + p(t)\dot{y} + q(t)y = \tfrac{1}{4}(n+3)\dot{y}^2 y^{-1} + \beta\exp(-2F)y^n \qquad \beta \neq 0, n \neq 1 \tag{2.5}$$

is given by the function

$$y = (au^2 + 2buv + cv^2)^{2/(1-n)} \qquad ac - b^2 = (1-n)\beta/4C_w^2 \tag{2.6}$$

where a, b and c are constants, two of which are arbitrary, and u and v are two linearly independent solutions of (2.2) where $\mu = (n+3)/4$.

It is seen here that the solutions given earlier by Eliezer and Gray [5] for (2.5) for the case $n = -3$ follows from (2.6) where C_w is given by (2.4).

Corresponding to the case $\beta = 0$ in (2.5), the result is given by the following theorem.

Theorem 2. If u and v are two linearly independent solutions of (2.2), the complete solution of the class of equations

$$\ddot{y}+p(t)\dot{y}+q(t)y=\mu\dot{y}^2y^{-1} \qquad \mu\neq 1 \tag{2.7}$$

is $y=(Au+Bv)^{1/(1-\mu)}$ where A and B are arbitrary constants. We will be using these results in our subsequent discussions.

3. Solutions

Let $u(r)$ and $v(r)$ be two linearly independent real functions and let

$$U(r)=u(r)+\mathrm{i}v(r) \tag{3.1}$$

be a solution of the linear differential equation

$$L_{1/2}R=0 \tag{3.2}$$

which is the radial Schrödinger equation whose coefficients are real functions of r. Hence we find that $\bar{U}(=u-\mathrm{i}v)$, u and v are also solutions of (3.2). Let $W(u,v)$ be the Wronskian and C_w the corresponding Abel constant. Abel's result for this case is

$$W(u,v)\exp(F(r))=C_w \qquad F(r)=2\log r. \tag{3.3}$$

From the definition $J_r=(\mathrm{i}\hbar/2m)[U\bar{U}'-\bar{U}U']$ where the primes denote differentiation with respect to r and the relations (3.1) and (3.3) we obtain

$$\begin{aligned} r^2J_r &= -C_w\hbar/m \\ J_r^2 &= C_w^2\hbar^2\exp(-2F)m^{-2}. \end{aligned} \tag{3.4}$$

Substituting for J_r, equation (1.2) becomes

$$L_1R=\tfrac{1}{2}R'^2R^{-1}+2C_w^2\exp(-2F)R^{-1}. \tag{3.5}$$

This equation is of the form (2.5) where $n=-1$ and $\beta=2C_w^2$. The corresponding associated equation is (3.2). Hence, from theorem 1, the general solution of Kostin's equation (3.5) is

$$R=au^2+2buv+cv^2 \qquad ac-b^2=1 \tag{3.6}$$

where a, b and c are constants, two of which are arbitrary. Since $R>0$ and u and v are real, it follows that the constants in (3.6) must be real.

Again we find that Kostin's equation (1.4) is in the form (2.7) where $\mu=\frac{1}{2}$. The corresponding associated equation is also (3.2) whose linearly independent real solutions are u and v. Hence the general solution of (1.4) is

$$R=(Au+Bv)^2 \tag{3.7}$$

where A and B are real arbitrary constants. This solution had been obtained earlier by Burt and Reid [6].

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